MATH 2050 - Intervals

(Reference : Bartle § 2.5)

We now discuss some nice subsets of $\mathbb{R}$ called "intervals".

$\exists$ 9 types of intervals (closed / open , bdd / unbdd)

Notation: Given $a, b \in \mathbb{R}$, $a < b$.

$(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$

$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$

$[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}$.

"bdd intervals"

$(a,\infty) := \{x \in \mathbb{R} \mid a < x\}$

$[a,\infty) := \{x \in \mathbb{R} \mid a \leq x\}$.

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

$(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$

$(-\infty, \infty) =: \mathbb{R}$

"unbdd intervals"

Def: Length$(I) := b - a > 0$.

Q: When is $S \subseteq \mathbb{R}$ an "interval" ?

A: "connectedness" (MATH 3070)

Thm: (Characterization of intervals)

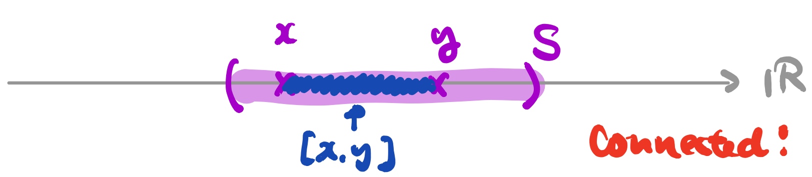
Let $S \subseteq \mathbb{R}$. Suppose

(i) $\exists$ $s_1, s_2 \in S$ s.t. $s_1 \neq s_2$

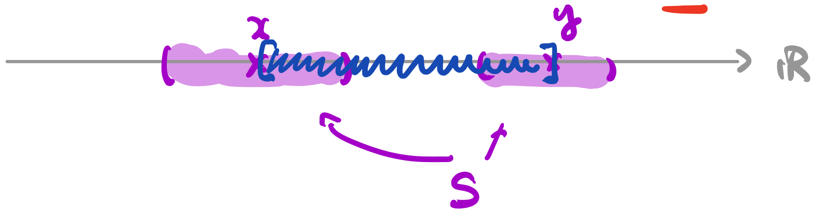
"Connected" *(ii) If $x, y \in S$, $x < y$, then $[x,y] \subseteq S$.

Then, S is an interval. [**Note:** could be unbdd.]

Picture:

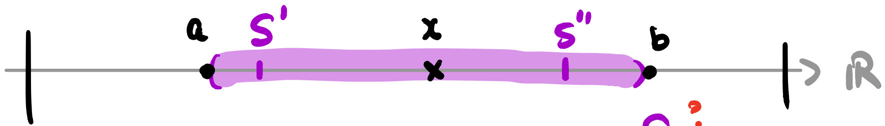


NOT connected!



Proof: We just treat the case $S \subseteq \mathbb{R}$ is bdd.

Picture:



$S \overset{?}{=} [a,b]$ or (a,b) or $[a,b)$ or $(a,b]$

Completeness Property $\Rightarrow$ $a := \inf S$, $b := \sup S$ exist in $\mathbb{R}$

By (i), we have $a \le s_1 < s_2 \le b \Rightarrow a < b$.

Claim: $(a,b) \subseteq S$

Pf of Claim: Take any $x \in (a,b)$, ie $a < x < b$

Want to show: $x \in S$.

Since $x > a = \inf S$, it cannot be a lower bd of S.

i.e. $\exists\ s' \in S$ s.t. $s' < x$

Since $x < b = \sup S$, it cannot be an upper bd of S

i.e. $\exists\ s'' \in S$ s.t. $x < s''$

By (ii), $[s', s''] \subseteq S$ but $x \in [s', s''] \Rightarrow x \in S$.

This implies $S = (a,b)$ or $[a,b)$ or $(a,b]$ or $[a,b]$.

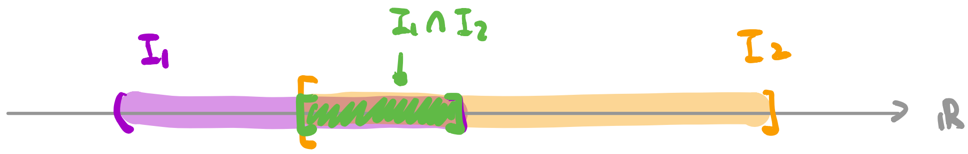
depending on whether $\inf S = a \in S$ or $\sup S = b \in S$.

$\blacksquare$

(maybe empty or degenerate)

Note: $I_1, I_2 \subseteq \mathbb{R}$ intervals $\Rightarrow$ $I_1 \cap I_2$ is always an interval.

BUT $I_1 \cup I_2$ might NOT be.



Q: What about $\bigcap_{i=1}^{\infty} I_i$?

Thm: ("Nested Interval Property" NIP)

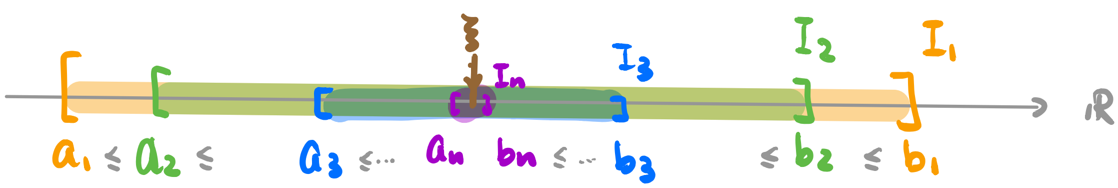
Let $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, be a seq. of closed and bounded intervals which are "nested":

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots\cdots\cdots$$

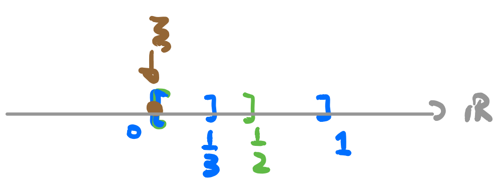
Then, $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Moreover, if $\inf\{\text{Length}(I_n) \mid n \in \mathbb{N}\} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Picture:



Examples: $\bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\}$



$\bigcap_{n=1}^{\infty} [0, 1+\frac{1}{n}] = [0,1] \neq \phi$.

Note that all conditions in the theorem cannot be dropped.

Non-examples:

(1) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$ not closed!

(2) $\bigcap_{n=1}^{\infty} [n, \infty) = \phi$ not bdd!

(3) $\bigcap_{n=1}^{\infty} [n, n+1] = \phi$ not nested!

Proof of Thm:

Recall: $I_n = [a_n, b_n]$, where $a_n \leq b_n$ $\forall n \in \mathbb{N}$.

Nested $\Rightarrow$ $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 - b_1$ $\forall n \in \mathbb{N}$

Consider $\phi \neq S := \{ a_n : n \in \mathbb{N} \} \subseteq \mathbb{R}$.

Note that S is bdd above since $a_n \leq b_1$ $\forall n \in \mathbb{N}$.

By Completeness Property, $\xi := \sup S \in \mathbb{R}$ exists.

Claim: $\xi \in \bigcap_{n=1}^{\infty} I_n$ (hence $\bigcap_{n=1}^{\infty} I_n \neq \phi$).

Pf of Claim: Want: $\xi \in I_n$ $\forall n \in \mathbb{N}$, ie. $a_n \leq \xi \leq b_n$

- $\xi = \sup S$ is an upper bd. $\Rightarrow$ $\xi \geq a_n$ $\forall n \in \mathbb{N}$
- To see why $\xi \leq b_n$ $\forall n \in \mathbb{N}$, we argue by contradiction.

Suppose NOT, ie. $\xi > b_m$ for some $m \in \mathbb{N}$

$\xi = \sup S \Rightarrow b_m$ is NOT an upper bd for S

$\Rightarrow \exists\, k \in \mathbb{N}$ st $b_m < a_k$

Contradiction !

Case 1: $m < k \Rightarrow b_k \le b_m < a_k \le b_k$

Case 2: $m \ge k \Rightarrow b_m < a_k \le a_m$

For the rest of the theorem, leave as exercise. □

Cor: $\mathbb{R}$ is uncountable.

Pf: It suffices to show $[0,1]$ is uncountable.

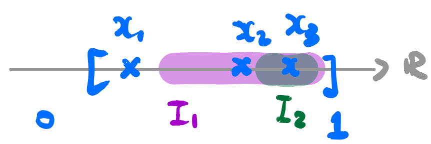
Argue by contradiction. Suppose $[0,1]$ is countable.

Then we can list them all into a sequence:

$$[0,1] = \{x_1, x_2, x_3, x_4, \dots\dots\} \quad \cdots\cdots \quad (*)$$

Define a seq of nested, closed, bdd intervals I_n, $n \in \mathbb{N}$ as follow:

- choose $I_1 \subseteq [0,1]$ s.t. $x_1 \notin I_1$
- choose $I_2 \subseteq I_1$ s.t. $x_2 \notin I_2$



....

- choose $I_n \subseteq I_{n-1}$ s.t. $x_n \notin I_n$

By NIP, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Suppose $\xi \in \bigcap_{n=1}^{\infty} I_n$.

$\Rightarrow \xi \in I_n \; \forall n \in \mathbb{N} \Rightarrow \xi \neq x_n \; \forall n \in \mathbb{N}$ Contradiction to $(*)$. $\xi \in [0,1]$ □